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SRM 4 Assig

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709
SRM - 4 assignment - 1.

Q1)
i. $Y_t^2 - \beta_1 e_t^2 Y_{t-1}^2 = 2(Y_t - \beta_1 e_t^2 Y_{t-1})\mu - (1-\beta_1 e_t^2)\mu^2 + \beta_0 e_t^2$

$Y_t^2 - 2Y_t\mu + \mu^2 = e_t^2(\beta_0 + \beta_1 Y_{t-1}^2 - 2\beta_1 Y_{t-1}\mu + \beta_1\mu^2)$

$\therefore (Y_t - \mu)^2 = e_t^2(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)$

$Y_t = \mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}$

$E(Y_t) = \mu + 0 \times E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})$

$= \mu$

$\text{Cov}(Y_t, Y_{t-1}) = \mu^2 - \mu^2$

$= 0$

ii. $\text{var}(Y_t | Y_{t-1}) = V(e_t) \cdot \text{var}(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)$

$= \beta_0 + \beta_1(Y_{t-1} - \mu)^2$

$\therefore$ var of Y_t is dependent on Y_{t-1}
& var of Y_t will depend on Y_{t-s}

$\therefore Y_t$ & Y_{t-s} are dependent

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iii. $\Delta X_t = X_t - X_{t-1}$

$E(\Delta X_t) = E(X_t) - E(X_{t-1})$

$= 0.5u + 0.3t + 0.5 - 0.5u - 0.3(t-1) - 0.1$

$= 0.3$

$\therefore$ constant

$\mathrm{cov}(\Delta X_t, \Delta X_{t-s}) = \mathrm{cov}(X_t - X_{t-1}, X_{t-s} - X_{t-s-1})$

$= \mathrm{cov}(0.3 + Y_t - Y_{t-1}, 0.3 + Y_{t-s} - Y_{t-s-1})$

$= \mathrm{cov}(Y_t, Y_{t-s}) - \mathrm{cov}(Y_t, Y_{t-s-1})$

$- \mathrm{cov}(Y_{t-1}, Y_{t-s}) + \mathrm{cov}(Y_{t-1}, Y_{t-s-1})$

$= 0$

$\therefore$ ACF is constant

Q2

i. characteristic eqn: $1 - z - 0.5z^2 + 0.5z^3 = 0$

rewrite in terms of $X = (1-B)Y$.

$X_t - 0.5X_{t-2} = Z_t + 0.3Z_{t-1} \Rightarrow \text{ARIMA}(2,1)$

$\therefore Y = \text{ARIMA}(2,1,1)$.

ii. characteristic polynomial of $X = (1 - 0.5z^2)$

whose roots are $\pm\sqrt{2}$

both roots > 1

$\therefore$ stationary

iii. model equation : $X_t = 0.5X_{t-2} + Z_t + 0.3Z_{t-1}$

$\mathrm{cov}(X_t, Z_t) = \sigma^2$

$\mathrm{cov}(X_t, Z_{t-1}) = 0.3\sigma^2$

$\mathrm{cov}(X_t, Z_{t-2}) = 0.5\sigma^2$

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$\gamma(0) = cov(X_t, X_t)$
$= 0.5\gamma(2) + 1.09\sigma^2$ — (1)

$\gamma(1) = 0.5\gamma(1) + 0.3\sigma^2$ — (2)
$\gamma(2) = 0.5\gamma(0)$ — (3)
$\gamma(k) = 0.5\gamma(k-2), \quad k>2$ — (4)

substituting (3) in (1)
$\gamma(0) = 0.25\gamma(0) + 1.09\sigma^2 \Rightarrow \gamma(0) = \frac{109}{75}\sigma^2$
from (2)

$\gamma(1) = \frac{3\sigma^2}{5}$

$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{45}{109}$

from (3) & (4)
$\rho(k) = 0.5^{|k|/2}$ if $|k|$ is even
$= \frac{45}{109} \times 0.5^{(|k|-1)/2}$ if $|k|$ is odd

Q3
i.
(a) Process is $(1 - 0.4B - 0.2B^2)Y_t$
$= (1+0.025B)Z_t + 0.016$

ch. eq^n $\Rightarrow$ $1 - 0.4z - 0.2z^2 = 0$
$\because$ no root having magnitude $=1$
$\therefore d = 0$
$\therefore$ ARIMA (2,0,1) process

(b) $(1-0.4-0.2)\,E(Y_t) = 0.016$

$\therefore E(Y_t) = 4\%$.

(c) 2 roots are $-1 \pm \sqrt{6}$

$= 1.4495$

& -3.4495

both are >1

$\therefore Y_t$ is a stationary Process

(ii)

(b) Inspection of the graph of time-plot of residuals: pattern such as clusters or uneven fluctuations which indicate inadequate fit.

Inspection of sample autocorrelation fn of the residuals: if too many residuals lie beyond range $+2/\sqrt{N}$ indicates poor fit or too few parameters.

Q4//

i. AR (2) process.

model can be written as : $\cancel{(1+\alpha B)}\cdot(1+\alpha B - \alpha^2 B^2)Y_t = Z_t$

$\therefore$ characteristic eq^n = $1 + ax - a^2x^2 = 0$

$$\therefore x = \frac{1 \pm \sqrt{5}}{2\alpha}$$

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$\therefore |\alpha| > 1$

$\therefore \dfrac{\sqrt{5}-1}{2|\alpha|} > 1$

$\dfrac{\sqrt{5}+1}{2|\alpha|} > 1$

$\therefore |\alpha| < \dfrac{\sqrt{5}-1}{2}$

ii.

$Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$

Yule walker eqn

$\gamma_0 = -\alpha\gamma_1 + \alpha^2\gamma_2 + \sigma^2$ — (1)

$\gamma_1 = -\alpha\gamma_0 + \alpha^2\gamma_1$ — (2)

$\gamma_2 = -\alpha\gamma_1 + \alpha^2\gamma_0$ — (3)

$\therefore \gamma_1 = -\dfrac{\alpha\gamma_0}{1-\alpha^2}$ — (4)

$\therefore \gamma_2 = -\alpha\left[-\dfrac{\alpha\gamma_0}{1-\alpha^2}\right] + \alpha^2\gamma_0$

$= (2\alpha^2 - \alpha^4)\dfrac{\gamma_0}{1-\alpha^2}$ — (5)

$= \gamma_0 = \dfrac{\sigma^2(1-\alpha^2)}{1-2\alpha^2-2\alpha^2+\alpha^6}$

& $\gamma_1 = \dfrac{\sigma^2(-\alpha')}{1-2\alpha^2-2\alpha^4+\alpha^6}$, $\gamma_2 = \dfrac{\sigma^2(2\alpha^2-\alpha^4)}{1-2\alpha^2-2\alpha^4+\alpha^6}$

Q5

i. $(x_n - x_{n-1}) = \alpha(x_{n-1} - x_{n-2}) + \varepsilon_n$.

$$\alpha = \rho_1 = \sum_{i=3}^{200} (x_i - x_{i-1})(x_{i-1} - x_{i-2}) \Big/ \sum_{i=2}^{200} (x_i - x_{i-1})^2$$

$$\Rightarrow \alpha = \frac{587.83}{936.49}$$

$$= 0.6277.$$

$$\gamma_0 = \frac{1}{200} \sum_{i=2}^{200} (x_i - x_{t-1})^2$$

$$= \frac{936.49}{200} = 4.6825$$

$$\therefore \gamma_0 = \frac{\sigma^2}{1-\rho_1^2} = \frac{\sigma^2}{1-\alpha^2}$$

$$\therefore \sigma^2_{hat} = (1 - \alpha^2_{hat})\, \gamma_0\ hat$$

$$= (1 - 0.6277^2) \times 4.6825$$

$$= 2.8376$$

ii. Forecast of x_{201} is:

$$(x_{200_{hat}}(1) - x_{200}) = 0.6277(1.93 - 0.82)$$

$$= 0.6967$$

$$\therefore x_{200_{hat}} = 2.6267$$

Q6

i. The purpose of a practical time series analysis may be summarised as follows:

- description of the data
- construction of a model which fits the data
- forecasting future values of the process
- for vector time series, investigating connections between 2 or more observed processes with the aim of using values of some of the processes to predict those of the others
- deciding whether the process is out of control, requiring action.

ii. AR(1) & random walk are Markov
MA(1) & AR(2) : not markov

iii.

a. $E[X(t)] = a + bt + E(Y(t))$

$\because Y(t) \Rightarrow$ stationary

$E(Y(t)) = c$ doesn't depend on t

$\therefore E(X(t)) = a + bt + c \Rightarrow$ depends on t

$\therefore X(t) \Rightarrow$ not stationary

b. $E[\nabla X(t)] = E(X(t)] - E[X(t-1)]$

$= a + b + c - [a + b(t-1) + c]$

$= b$

$\therefore$ mean doesn't depend on t

$\text{cov}[\nabla X(t), \nabla X(s)] = C_Y(t-s) - C_Y(t-s+1) - C_Y(t-s-1) + C_Y(t-s)$.

as $\nabla X(t)$ is stationary but $X(t)$ is not the process, the process $X(t)$ is $I(1)$

(iv) $$X_1(t) - X_2(t) = a[X_1(t-1) - X_2(t-1)] + b[X_2(t-1) - X_1(t-1)] + [e_1(t) - e_2(t)]$$

$$= (a-b)[X_1(t-1) - X_2(t-1)] + [e_1(t) - e_2(t)]$$

$\because e_1(t)$ & $e_2(t)$ are WNP

$\therefore e(t) = $ WNP

$\therefore Y(t) = AR(1)$ process.

$Y(t)$ is stationary as long as $|a-b| < 1$.

Q7

i.

a. AR : An auto regressive process with order p is a sequence of random variables (X_t) defined by the rule:

$$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \cdots + e_t$$

$\therefore$ AR(p) model attempts to explain the current value X as a linear combination of past values with some additional externally generated random variable.

b. MA process :

$$X_t = \mu + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}$$

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MA(q) explains the relation between X_t as an indirect effect, arising from the fact that the current value of the process results from past & current error terms.

c. ARMA:

$$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \cdots + \alpha_p (X_{t-p} - \mu) + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}$$

ii. (a) It is MA(1) process & hence stationary
Also, ARIMA (0, 0, 1)

(b) ARIMA (2,3) process.
To classify as ARIMA (2, 0, 3), we check for stationarity:

$$(1 - 1.4B^2) X_t = e_t + 0.5e_t$$

$$\phi(\lambda) = 1 - 1.4\lambda^2 = 0$$

$$\lambda = \pm 0.8452$$

∴ both roots < 1

∴ non stationary

(c) ARIMA (2,1) process:

$$\nabla X_t = 0.4 \nabla X_{t-1} = e_t + e_{t-1}$$

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Q8

i. $X_t = A_t + B_t$

$A_t = 0.5A_{t-1} + 0.5B_{t-1} + e_t^{(A)}$

$B_t = 0.7A_{t-1} - 0.7A_{t-2} + e_t^{(B)}$

$$\therefore Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} Y_{t-1} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} Y_{t-2} + \begin{pmatrix} e_t^{(A)} \\ e_t^{(B)} \end{pmatrix}$$

$\therefore \lambda^2 - 0.5\lambda - 0.35 = 0$

$|\lambda| < 1$

similarly for second matrix

$|\lambda| < 1$

$\therefore Y_t \Rightarrow$ stationary

ii.

a. $X_t = (\alpha+1)X_{t-1} - (\alpha + 0.25\alpha^2)X_{t-2} + 0.25\alpha^2 X_{t-3} + e_t$

$Y_t = \alpha Y_{t-1} - 0.25\alpha^2 Y_{t-2} + e_t$

$[Y_t = X_t - X_{t-1}]$

$\therefore X_t \Rightarrow ARIMA(2,1,0)$

$\therefore (1 - \alpha B + 0.25\alpha^2 B^2) Y_t = e_t$

ch. eq $= 1 - \alpha\lambda + 0.25\alpha^2\lambda^2 = 0$

$\therefore \lambda = \frac{2}{\alpha}$

for $|\lambda| > 1 \Rightarrow |\alpha| < 2$

$\therefore X_t \Rightarrow ARIMA(2,1,0)$ when $|\alpha| < 2$

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b. $cov(Y_t, e_t) = cov(e_t, e_t) = \sigma^2$

$\gamma_0 = cov(Y_t, Y_t) = \alpha \gamma_1 - 0.25\alpha^2 \gamma_2 + \sigma^2$ — (1)

take covariance:

$\therefore \gamma_1 = \alpha\gamma_0 - 0.25\alpha^2\gamma_1$ — (2)

& $\gamma_2 = \alpha\gamma_1 - 0.25\alpha^2\gamma_0$ — (3)

$$\gamma_k = \alpha\gamma_{k-1} - 0.25\alpha^2\gamma_{k-2}$$

$$\therefore \gamma_1 = \frac{\alpha\gamma_0}{1+0.25\alpha^2}$$

$$\gamma_0 = \alpha\gamma_1 - 0.25\alpha^2(\alpha\gamma_1 - 0.25\alpha^2\gamma_0) + \sigma^2$$

$$\therefore \gamma_0 = \frac{1+0.25\alpha^2}{(1-0.25\alpha^2)^3} \cdot \sigma^3$$

$$\& \ \gamma_1 = \frac{\alpha}{(1-0.25\alpha^2)^3} \cdot \sigma^2.$$

for $k \geq 2$, $\gamma_k = \alpha\gamma_{k-1} - 0.25\alpha^2\gamma_{k-2}$

$$\therefore \lambda_1 + k\lambda_2 = (0.5\alpha)^{-k} \cdot \gamma_k.$$

$$\therefore \gamma_k = \lambda_1 \alpha^k (0.5)^k.$$

$$\therefore \lambda_1 = \gamma_0 = \frac{(1+0.25\alpha^2)}{(1-0.25\alpha^2)^3} \cdot \sigma^2$$

$$\lambda_2 = \frac{1}{0.5\alpha}\gamma_1 - \lambda_1$$

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c. $\alpha = 0.04$

$\therefore Y_t = 0.04\, Y_{t-1} - 0.0004\, Y_{t-2} + e_t$

$Y_t = X_1 - X_{t-1}$

$X_t = Y_t + X_{t-1}$

$\because X_1, X_2, \ldots, X_{50}$ are obs value

$X_{51} = Y_{51} + X_{50}$

$X_{52} = Y_{52} + X_{51}$

$\therefore Y_{51} = 0.04\,(X_{50} - X_{49}) - 0.0004\,(X_{49} - X_{48})$

& $Y_{52} = 0.04\, Y_{51} - 0.0004\,(X_{50} - X_{49})$

Q10 i. $p = 1$

$q = 4$

$\therefore$ ARMA (1,4)

11. non-linear, non-stationary time series models includes -

Bilinear: $X_n - \alpha(X_{n-1} - \mu)$

$= \mu + e_n + \beta e_{n-1} + b\,(X_{n-1} - \mu)\, e_{n-1}$ [Bursty behaviour]

Threshold AR: $X_n = \mu + \alpha_2 (X_{n-1} - \mu) + e_n, \; X_{n-1} \le d$ [cyclical behaviour]

Random coeff AR: $X_t = \mu + \alpha(X_{t-1} - \mu) + e_t$

ARCH models are used for asset prices:

$X_t = \mu + e_t \sqrt{\alpha_0 + \sum_{k=1}^{p} \alpha_k (X_{t-k} - \mu)^2}$

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Q11.

i. purpose of practical Time series analysis:
- description of data
- deciding whether process is out of control, requiring action.

ii. univariate time series
↳ sequence of observations recorded at regular time intervals.
↳ continuous state space & discrete time set.

Invertibility
↳ invertible time series process if we can write white noise terms as a convergent sum of the x terms.
↳ desirable characteristic ∵ it enables us to calculate the residual terms and analyse good fit of the model.

Markov
↳ $P[X_t = \alpha \mid X_{s_1} = x_1, \dots, X_{s_n} = x_n, X_s = x]$
$= P[X_t = \alpha \mid X_s = x]$
for all times, $s_1 < s_2 < \dots < s_n < s < t$ & all states: $\alpha, x_1, \dots x_n$ of S.
↳ we can predict the future state from current state alone

iii. To remove seasonal variation:
- seasonal differencing.
- method of moving average

To remove linear trend
- least square trend removal
- differencing.

IV.

a. $y_t - 0.6y_{t-1} = 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}$

$= y_t = 1.25 + 0.2e_t + 0.19e_{t-1} + 1.192 \sum_{t=2}^{\infty} 0.6^{i-2} e_{t-i}$

b. now.

$cov(y_t, e_t) = 0.2\sigma^2$

$cov(y_t, e_{t-1}) = 0.82\sigma^2$

$cov(y_t, e_{t-2}) = 0.792\sigma^2$

$cov(y_{t-1}, e_{t-2}) = 0.82\sigma^2$

$\gamma_0 = 0.6\gamma_1 + 0.8516\sigma^2$

$\gamma_1 = 0.6\gamma_1 + 0.66\sigma^2$

$\therefore \gamma_0 = 1.6925\sigma^2$

$\gamma_1 = 1.4015\sigma^2$

$\gamma_2 = 0.9009\sigma^2$

$\& \gamma_k = 0.6\gamma_{k-1}, \quad k \geq 3$

$\rho_0 = 1$

$\rho_1 = 0.8281$

$\rho_2 = 0.5323$

$\rho_k = 0.6\rho_{k-1}, \quad k \geq 3.$

Q17.

i. Lack of stationarity :
- deterministic trend or cycle.
- integrated version
- eg: company selling greeting cards will see a hike in sells in particular months

ii.

(a) $(\nabla^d x_t - \mu) = \sum^{p} \alpha_i (\nabla^d x_{t-1} - \mu) + e_t + \sum_{1}^{q} \beta^j e_{t-j}$
is a form of ARIMA (p, d, q) process

(b) main steps in box-jenkins methodology :
- Tentative identification of a model from ARIMA class
- Estimation of the parameter in the identified Model
- Diagnostic checks

(c.) if sample auto correlation coefficients decay slowly from 1 then this indicates that further differencing is required. This is not the case for d = 1, which means that the differencing of the original series is required once, hence d = 1
correct value of d minimises the sample variance. This also indicates that d = 1.

d.

i. $X_t = 0.8e_{t-1} + e_t$ ∵ MA(1) Process & stationary

∴ ARIMA (0,0,1)

ii. $X_t = 2X_{t-2} + e_t + 0.5e_{t-3}$ ∴ ARIMA (2, 3) process. cannot be differenced

∴ ARIMA (2,0,3)

check for stationarity = $(1-2B^2)X_t = e_t + 0.5e_{t-3}$

ch. eqn = $\phi(\lambda) = 1 - 2\lambda^2 = 0$

$\lambda = \pm 1/\sqrt{2}$

∵ $|\lambda| < 1$

not stationary

∴ ARMA (2,3)

iii. $X_t = 1.5X_{t-1} + 0.2X_{t-2} + e_t + e_{t-1}$: ARMA (2,1)

differencing gives : $X_t - 1.5X_{t-1} - 0.2X_{t-2} = e_t + e_{t-1}$

$= (X_t - X_{t-1})$

Q13 i. weak stationary if it has constant mean covariance is fixed for each lag [fixed lag]

ii. $X_n = Z_n + \beta Z_{n-1}$

$E[X_n] = (1+\beta)\mu \Rightarrow$ constant

$V(X_n) = V(Z_n + \beta Z_{n-1})$

$= V(1+\beta^2)\sigma^2$

$cov(X_n, X_{n+1}) = \beta\sigma^2$

for $m > 1$ = $cov(X_n, X_{n+1}) = 0$

$\therefore$ process = weakly stationary.

iii. model eqn: $(1 - 1.5B + .5B^2)x = Z$

$\therefore (1-B)(1-0.5B)$

magnitude = 1

$\therefore$ non stationary

iv. ARIMA (1,1,0)

$\therefore (1-B)x$ = AR(1)

& $Y = (1-B)x$

$\therefore (1-0.5B)Y = Z$: AR(1)

$\therefore$ autocorrelation at lag $1 = 0.5$