Assignment 2

23 March 2022 16:24

Q1

(i)

$$\begin{array}{c|cc} & B & O \\ \hline B & -\lambda & \lambda \\ O & \mu & -\mu \end{array}$$

$$\begin{array}{cc} & \begin{matrix} B & O \end{matrix} \\ \begin{matrix} B \\ O \end{matrix} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

(ii) $B \longrightarrow \exp(\lambda_i)$

$O \rightarrow \exp(\mu_i)$

(iii)

$$\frac{d}{dt}\, {}_tP_s^{BB} = {}_tP_s^{BB}(-\lambda_t) + {}_tP_s^{BO}(\mu_t) \quad (1)$$

$$\frac{d}{dt}\, {}_tP_s^{BO} = {}_tP_s^{BO}(-\mu_t) + {}_tP_s^{BB}(\lambda_t)$$

(iv) $p^{BB} + p^{BO} = 1$

(iv) ${}_tP_s \quad {}_tP_s$

substituting in (i)

$$\frac{d}{dt}\,{}_tP_s^{BB} = -\lambda\,{}_tP_s^{BB} + \mu\left(1 - {}_tP_s^{BB}\right)$$

$$\frac{d}{dt}\,{}_tP_s^{BB} = -\lambda\,{}_tP_s^{BB} + \mu - \mu\,{}_tP_s^{BB}$$

$$\frac{d}{dt}\,{}_tP_s^{BB} = -{}_tP_s^{BB}(\lambda+\mu) + \mu$$

$$\Rightarrow \frac{d}{dt}\,{}_tP_s^{BB} + (\lambda+\mu)\,{}_tP_s^{BB} = +\mu$$

$$I.F. = e^{\int(\lambda+\mu)\,dt}$$

$$\Rightarrow \frac{d}{dt}\left[\exp\left((\lambda+\mu)t\right)\cdot{}_tP_s^{BB}\right] = \mu\cdot\exp\left((\lambda+\mu)t\right)$$

$$\Rightarrow \exp\left((\lambda+\mu)t\right)\cdot{}_tP_s^{BB}$$

$$= \frac{\mu}{\lambda+\mu} \exp\left((\lambda+\mu)t\right) + c$$

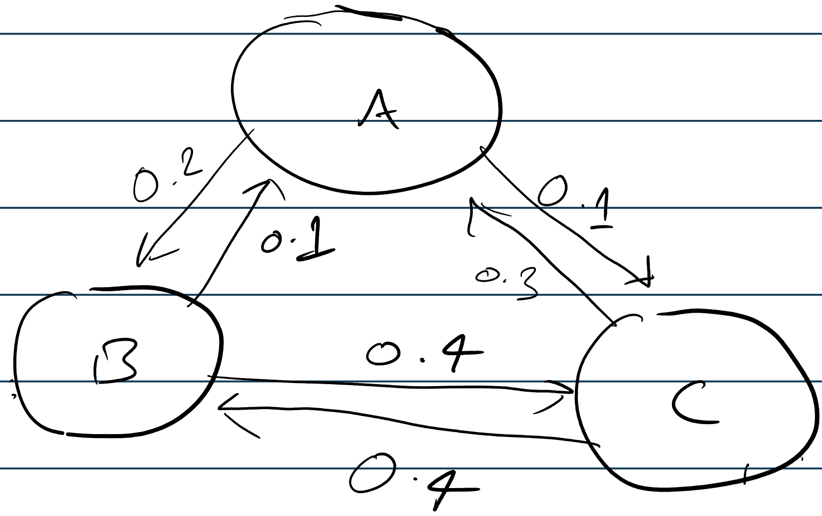
$$\text{Constant} = \frac{\lambda}{\lambda+\mu}$$

$$\epsilon\, P_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \exp\left(-(\lambda+\mu)t\right)$$

q2)

(i)

$$\begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} -0.3 & 0.2 & 0.1 \\ 0.1 & -0.5 & 0.4 \\ 0.3 & 0.4 & -0.4 \end{pmatrix} \end{array}$$



(ii) $\frac{d}{dt} P(t) = -0.2\, P(t) + 0.1\, P(t)$

$$\frac{d}{dt} P_{AA}(t) = -0.3 P_{AA}(t) + 0.1 P_{AB}(t) + 0.3 P_{AC}(t)$$

$$\frac{d}{dt} P_{AB}(t) = 0.2 P_{AA}(t) - 0.5 P_{AB}(t) + 0.1 P_{AC}(t)$$

$$\frac{d}{dt} P_{AC}(t) = 0.1 P_{AA}(t) + 0.4 P_{AB}(t) - 0.4 P_{AC}(t)$$

(iii) $\frac{d}{dt} P_{\overline{AA}}(t) = -0.3 P_{\overline{AA}}(t)$

$P_{\overline{AA}}(t) = \exp(-0.3t)$

for $t=2$,

$\exp(-0.6) = 0.5488$

(iv) $BAC = \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} = \frac{2}{45}$

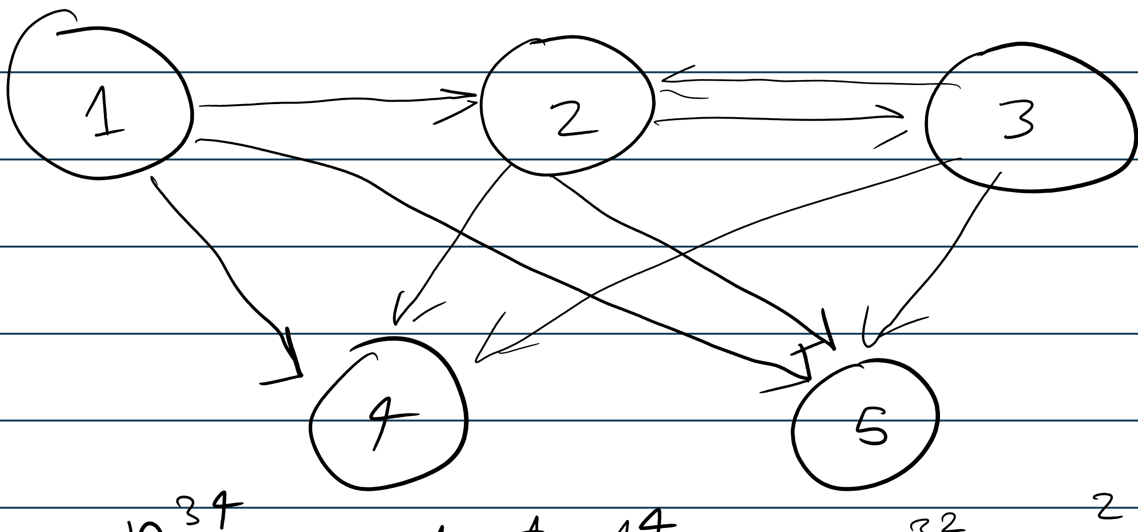
$CAC = \frac{1}{} \times \frac{3}{} \times \frac{1}{} = \frac{1}{}$

$CAC = \frac{1}{3} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{12}$

$CBC = \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} = \frac{1}{15}$

$Sum = \frac{7}{36} = 0.194$

3(i) 1) Never taken Nimble
2) Taking Nimble
3) No longer taking Nimble
4) Death by heart disease
5) Death due to other causes



(ii) ${}_{t+dt}P_x^{34} = {}_tP_x^{31}\,{}_{dt}P_{x+t}^{14} + {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24}$

$$+\,{}_tp_x^{33}\,{}_{dt}p_{x+t}^{34} + {}_tp_x^{34}\,{}_{dt}p_{x+t}^{44} + {}_tp_x^{35}\,{}_{dt}p_{x+t}^{54}$$

$${}_tp_x^{31} = {}_{dt}{}_{x+t}^{54} = 0$$

$${}_{dt+t}p_x^{34} = {}_tp_x^{32}\,{}_{dt}p_{x+t}^{24} + {}_tp_x^{33}\,{}_{dt}p_{x+t}^{34}$$
$$+\,{}_tp_x^{34}\,{}_{dt}p_{x+t}^{44}$$

$$\therefore\ {}_{dt}p_{x+t}^{44} = 1$$

$${}_{dt}p_{x+t} = \mu_{x+t}^{ij}\,dt + o(dt)$$

$$\lim_{dt \to 0} \frac{o(dt)}{dt} = 0,$$

$${}_{dt+t}p_x^{34} = {}_tp_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tp_x^{33}\,\mu_{x+t}^{34}\,dt$$
$$+\,{}_tp_x^{34} + o(dt)$$

$${}_{dt+t}p_x^{34} - = {}_tp_x^{32}\,\mu_{x+t}^{24}\,dt$$

$${}_{dt+t}p_x \ldots {}_tp_x\,\mu_{x+t}\,dt$$

$$+ {}_tp_x^{33}\,\mu_{x+t}^{34}\,dt + {}_tp_x^{34} + O(dt)$$

so that,

$${}_{dt+t}p_x^{34} - {}_tp_x^{34} = {}_tp_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tp_x^{33}\,\mu_{x+t}^{34}\,dt$$

$$+ O(dt)$$

4)(i) $X \sim Poi(3)$

Expected number of calls = 3

(ii) Poisson process is memoryless

Expected number of calls = 3/60 mins

$\therefore$ $\boxed{\text{Next call in 20 mins}}$

(iii) Expected number of calls in

0.5 hr = 1.5

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{\cdot}$$

$x!$

$$\Rightarrow P(X=0) = \frac{e^{-1.5}(1.5)^0}{0!}$$

$$\Rightarrow \boxed{P(X=0) = 0.22313}$$

(iv) Expected number of calls per hour = 3

" duration of 1 call = 7 minutes

Duration of line engager per hour

= 3 × 7 = 21 mins

P(line is engaged when a caller calls)

$$= \frac{21}{60} = \boxed{0.35}$$

5) (i) Using the markov assumption,

$$p_{HH}(x, t+dt) = p_{HH}(x,t)\, p_{HH}(t, t+dt)$$

$$+ p_{HS}(x,t)\, p_{SH}(t, t+dt)$$

$$+ p_{HD}(x,t)\, p_{DH}(t, t+dt)$$

$$p_{DH}(t, t+dt) = 0$$

$\therefore$

$$p_{HH}(x, t+dt) = p_{HH}(x,t)\, p_{HH}(t, t+dt)$$
$$+ p_{HS}(x,t)\, p_{SH}(t, t+dt)$$

for small dt,

$$p_{ij}(t, t+dt) = \lambda_{ij}(t)\, dt + (o)\, dt$$

$$p_{ij}(t, t+dt) = \cancel{1} + \lambda_{ii}(t)\, dt + (o)\, dt$$

λ = instantaneous transition rates

$$\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$$

$$p_{HH}(x, t+dt) =$$

$p_{HH}(x,t)\left(1-\sigma(t)\,dt-\mu(t)\,dt\right)$

$+\ p_{HS}(x,t)\,\rho(t,C_t)+o(dt)$

Hence,

$$\frac{d}{dt}p_{HH}(x,t)=\lim_{dt\to 0}\frac{p_{HH}(x,t+dt)-p_{HH}(x,t)}{dt}$$

$$=p_{HH}(x,t)\left(-\sigma(t)-\mu(t)\right)+p_{HS}(x,t)\,\rho(t,C_t)$$

(ii) $\frac{d}{dt}\bar{p}_{HH}(0,t)=-\left(\sigma(t)+\mu(t)\right)\bar{p}_{HH}(t)$

Q6. All 3 processes have a discrete state space

- Markov chain and Markov Jump Chain operate in discrete time while Markov Jump process operates in continuous time
- All of them follow the Markov

Property

(future developement of the process can be predicted from the present state alone, without reference to past history.

- The Markov Chain and markov Jump Chain are expressed in terms of probabilities while the Markov Jump process is expressed in terms of transition rates.

Q7(i) The MLEs of the transition intensities from state i to state j is the number of transitions from state i to state j divided by total waiting time in state i

To estimate the transition intensities wee need —
- the total time spent in each state

OR
- entry and exit times for each individual in each state

· total no. of transitions of each type

(ii) $\frac{d}{dt} P_{AA}(s,t) = -P_{AA}(s,t)\,\mu$ -

$\frac{d}{dt} P_{AT}(s,t) = P_{AA}(s,t)\,\mu$

(iii) $\frac{1}{P_{AA}(t)} \frac{d}{dt} P_{AA}(t) = -\mu$

$\frac{d}{dt}\left(\ln\left(P_{AA}(t)\right)\right) = -\mu$

$\therefore P_{AA}(t) = \exp(-\mu t + c)$

$P_{AA}(0) = 1$

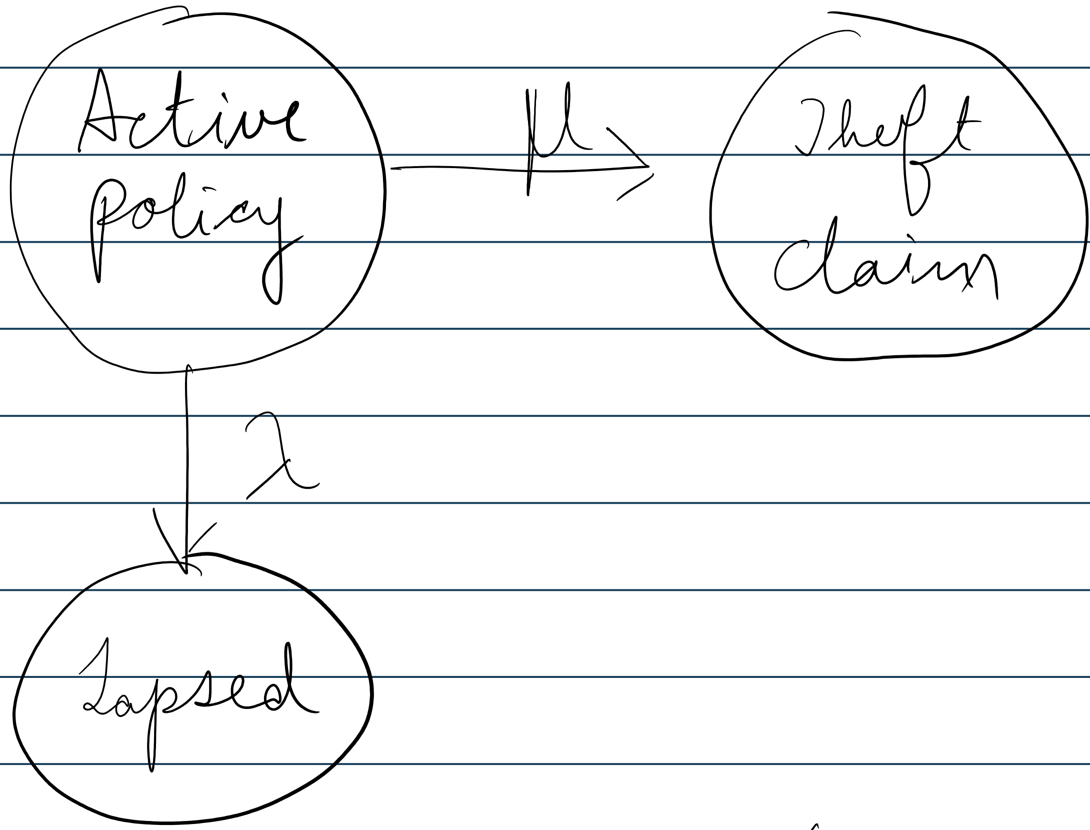
$\therefore c = 0$

$\therefore P_{AA}(t) = \exp(-\mu t)$

Expected Cost = $\exp(-\mu t)$

(iv)

(iv)



(v) $$\frac{d}{dt} p_{AA}(t) = -p_{AA}(t)(\mu+\lambda)$$

$$p_{AA}(t) = -\exp(-(\mu+\lambda)t)$$

We want,

$$\frac{d}{dt} p_{AT}(t) = p_{AA}(t)\mu = \mu \exp(-(\mu+\lambda)t)$$

$$\Rightarrow p_{AT}(t) = \left[\frac{-\mu}{(\mu+\lambda)} \exp(-(\mu+\lambda)t)\right]_0^T$$

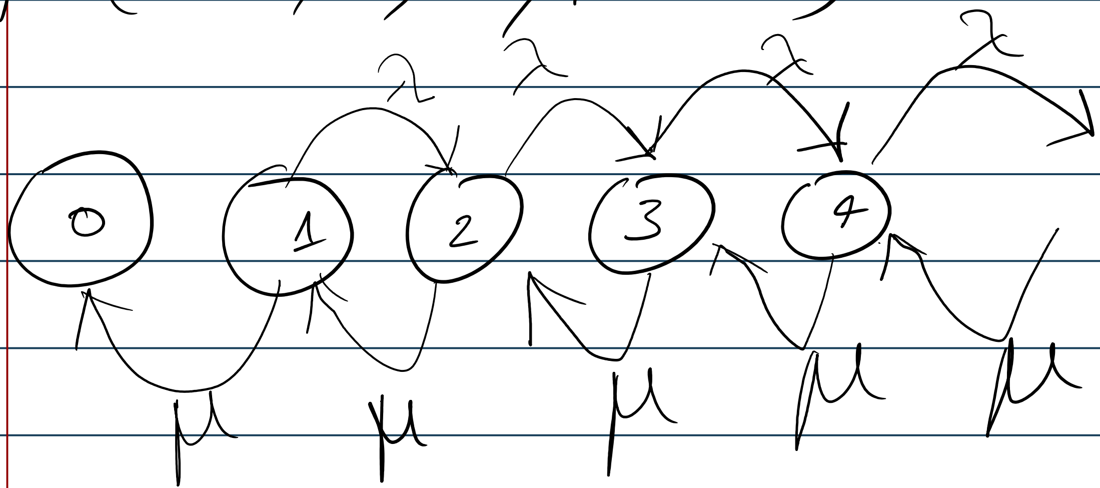
$$= \frac{\mu}{\mu+\lambda}\left(1-\exp(-(\mu+\lambda)T)\right)$$

$\mu+\lambda$

$$\text{Claims} = \frac{\mu}{\mu+\lambda} C\left(1-\exp\left(-(\mu+\lambda)T\right)\right)$$

q8(i) $(0, 1, 2, 3, 4 \ldots)$

(ii)



(iii)

$$\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & \mu & -(\mu+\lambda) & \lambda & 0 & 0 \\
2 & 0 & \mu & -(\mu+\lambda) & \lambda & 0 \\
3 & 0 & 0 & \mu & -(\mu+\lambda) & \lambda \\
4 & 0 & 0 & 0 & \mu & -(\mu+\lambda)
\end{array}$$

(iv) If a markov jump process X_t is examined only at the times of transition, the resulting process is called the Markov jump chain associated with x_t.

(v)

	0	1	2	3	4
0	0	0	0	0	0
1	$\frac{\mu}{\mu+\lambda}$	0	$\frac{\lambda}{\mu+\lambda}$	0	0
2	0	$\frac{\mu}{\mu+\lambda}$	0	$\frac{\lambda}{\mu+\lambda}$	0
3	0	0	$\frac{\mu}{\mu+\lambda}$	0	$\frac{\lambda}{\mu+\lambda}$
4	0	0	0	$\frac{\mu}{\mu+\lambda}$	0

(vi) $\left(\frac{\mu}{\mu+\lambda}\right)^3$

Q.10

(i) $$\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -2t \times P_{\bar{A}\bar{A}}(t)$$

$$\Rightarrow \frac{d}{dt}\left[\ln P_{\bar{A}\bar{A}}(t)\right] = -2t$$

or

$$\ln P_{\overline{AA}}(s) = -s^2 + \text{constant}$$

$$P_{AA}(0) = 1$$

$\therefore$ constant $= 0$

$$\therefore \; P_{\overline{AA}}(s) = \exp(-s^2)$$

(ii) P(in 1st visit to B at time T in state A at $t=0$)

$$= \int_0^T P(\text{remains in A to time } s)$$

$\times P$(transition to B in time $s, s+ds$)

$\times P$(remains in B to time T) ds

$$= \int_{s=0}^{T} P_{\overline{AA}}(s) \times 2s \times P_{\overline{BB}}(s,T)\, ds$$

$$= \int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2}\, ds$$

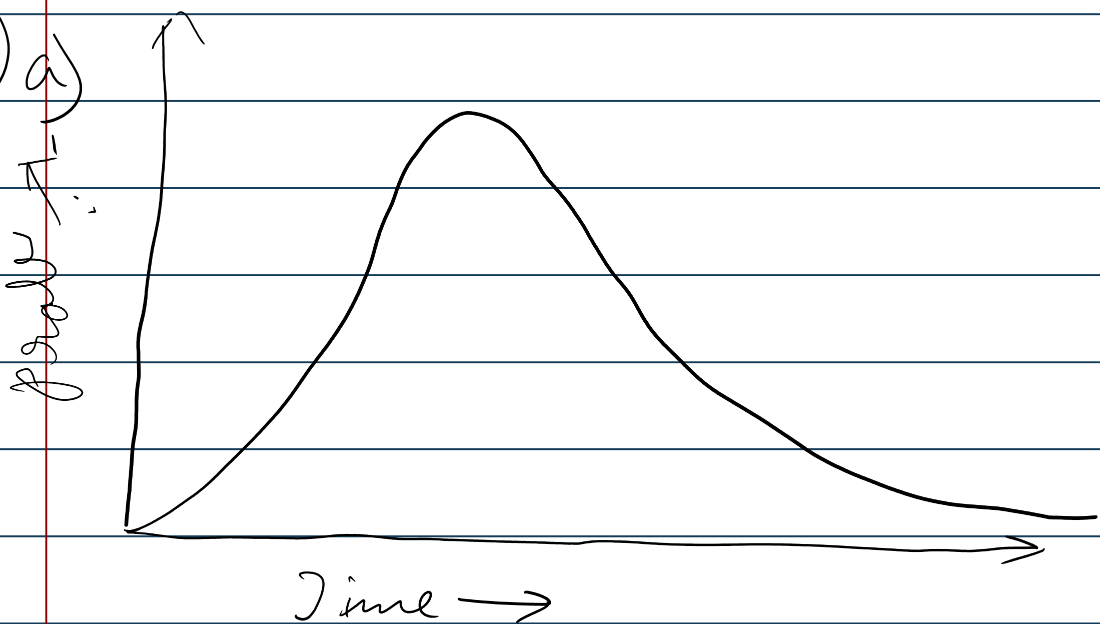
$$\int^{T} \quad e^{-T^2+s^2} \; d$$

$$= \int_{s=0}^{T} 2s \times e^{-T^2+s^2} \, ds$$

$$= \int_{s=0}^{T} 2s \times e^{-T^2} \, ds$$

$$= e^{-T^2} \times T^2$$

(iii) a)



b) Initially probability increases from 0 @ $T=0$ and accelerates as the transition rate from A to B increases.

However, as transitions increase, it becomes more likely that the process has already visited state B and jumped back to A. Therefore, the probability of being in the 1st visit to B tends (exponentially) to 0.

c) Differentiate to find turning point:

$$\frac{d}{dt}\left[e^{-t^2} \times t^2\right] = 2t \times e^{-t^2} - 2t^3 \times e^{-t^2}$$

equate to 0

$$e^{-t^2} \times 2t \times (1 - t^2) = 0$$

This is maxima.

Q11)

(i) Let N_t denote the no. of claims up to time t. Since the Poisson process has stationary increments, we may take $t = 0$ so,

$$P(T_0 \leq y \mid N_5 = 1) = \frac{P(T_0 \leq y, N_5 = 1)}{P(N_5 = 1)}$$

$$= P(N_4 = 1, N_5 - N_4 = 0)$$

$$= \frac{P(N_y = 1, N_s - N_y = 0)}{P(N_s = 1)}$$

$N_s - N_y$ is independent of N_y and has the same distribution as N_{s-y}.

$\therefore$ RHS =

$$\frac{\left(\lambda y e^{-\lambda y}\right) e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$$

which is the CDF of uniform dist. on $[0,s]$

(ii) $\because$ Holding times are independent and have an exponential distribution.

Joint density $\rightarrow$

$$\lambda^n e^{-\lambda(t_1 + t_2 + \dots + t_n)} 1_{\{t_1, t_2 \dots t_n > 0\}}$$

(iii) $P(N_s = k \mid N_t = n)$

$$= \frac{P(N_s = k, N_t = n)}{P(N_t = n)}$$

$$= \frac{P(N_s = k, N_t - N_s = n-k)}{P(N_t = n)}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^k}{k!} \cdot \dfrac{e^{-\lambda(t-s)}\lambda^{n-k}(t-s)^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{e^{-\lambda t}\lambda^n s^k (t-s)^{n-k}}{k!(n-k)!} \cdot \frac{n!}{e^{-\lambda t}\lambda^n t^n}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{s^k (t-s)^{n-k}}{t^k t^{n-k}}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

which is binomial with parameters n and $\frac{s}{t}$