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Probability & Statistics

① x_i — claim size of home insurance
y_i — claim size of car insurance.

$$X \sim N(800, 100^2)$$
$$Y \sim N(1200, 300^2)$$

- ~~$800 < P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$~~

~~$P(Y_1 + Y_2 + Y_3 - (X_1 + X_2 +$~~

- $800 < P(Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4)$

$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$.

$= (Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim$

$N(3 \times 1200 - 4 \times 800,),$

$= 300^2(3) + 100^2(4) \sim N(400, 310000)$

$= P\left(Z > 800 - 400 / \sqrt{310000}\right)$

$= P(Z > 0.71842)$

$= 1 - P(Z < 0.71842)$.

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(3) i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$

$MGf = E(e^{tx})$

$$= \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)}$$

$$M_x t = \lambda \int_{\alpha}^{\infty} e^{tx} \cdot e^{-\lambda x} \cdot e^{\lambda\alpha} dx$$

$$= \frac{e^{\lambda\alpha} \lambda}{1-\lambda} \left[e^{-(\lambda-t)x} \right]_{\alpha}^{\infty}$$

$$= \frac{e^{t\alpha} \cdot \lambda}{t-\lambda} \left[-e^{-(\lambda-t)\alpha} \right]$$

$$= \frac{e^{t\alpha} \lambda}{\lambda - t}$$

ii) $M_x t = \frac{e^{t\alpha} \lambda}{\lambda - t}$

$$= \lambda e^{t\alpha} (\lambda - t)^{-1}$$

$$M_x' t = \lambda [e^{t\alpha} (\lambda - t)^{-2} (1) + (\lambda - t)^{-1} e^{t\alpha} (\alpha)]$$

$$= \lambda [(\lambda - t)^{-1} e^{t\alpha} (\alpha) + e^{t\alpha} (\lambda - t)^{-2}]$$

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$= \lambda e^{t\alpha}(\lambda - t)^{-2}[\alpha(\lambda-t)+1]$

$= \lambda(\lambda)^{-2}[\alpha(\lambda)+1]$

$= \frac{1}{\lambda}[\lambda\alpha+1] = \lambda\alpha+1/\lambda$

$E(x) = M_x''(t) =$

$\lambda[\alpha e^{t\alpha}(\lambda-t)^{-2} + \alpha^2(\lambda-t$
$e^{t\alpha} + e^{t\alpha}(+2)$
$(\lambda-t)^{-3} + (\lambda-t)^{-2}e^{t\alpha}]$

$= \lambda[\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + (2)(\lambda)$
$+ \alpha(\lambda)^{-2}]$

$\lambda\left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3}\right]$

$= E(x)^2 = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$

$= V(x) = E(x)^2 - [E(x)]^2$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\alpha + \frac{1}{\lambda}\right]$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda}\right)$

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(4) $G_v(t) = \left(\frac{p}{1-qt}\right)^k$

$G_v t = \sum t^v \times P(V=v)$

$P(v=4) \to \frac{\sqrt{k+4}}{\sqrt{5}\sqrt{k}} \cdot p^k \cdot q^4$

$\frac{\sqrt{k+4}}{4!\sqrt{k}} p^k q^k$

$= \frac{(k+3)!}{(k-1)!\,4!} p^k (1-p)^k$

(5) .

i) $P(y > \frac{1}{3}x + 10)$

$9 \int_{10}^{20} \int_{0}^{5} (x^2 + xy)\, dx \cdot dy = 1$

$= \int_{10}^{20} \left(\frac{125}{3} + \frac{25}{2}y\right) \cdot dy$

$= \frac{1}{9} = \left(\frac{125}{3}y + \frac{25}{4}y^2\right)_{10}^{20}$

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$= 833.33 + 2500 - (416.66 + 625) = \frac{1}{9}$

$= 9 = 0.\cancel{000}\,43636$

iii) $f\,y/x = \frac{f(x, y)}{f(x)}$

$= f(x) = \frac{3}{6875} \int_{10}^{20} (x^2 + xy)\,dy$

$= \frac{3}{6875} \left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$

$= \frac{3}{6875} \left[20x^2 + x(200) - 10x^2 - 50x\right]$

$= \frac{3}{6875} \left[10x^2 + 150x\right]$

$= \frac{30}{6875} \left[x + 15\right]$

$f\,y/x = \frac{x(x+y)}{x} \times \frac{6875}{\cancel{(x+15)}} (x+15)$

$E(Y/X) = \int_{10}^{20} y \cdot \frac{(x+y)}{10(x+15)}\,dy$

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$$= \frac{1}{10(x+15)} \int_{10}^{20} (xy+y^2)\, dy$$

$$= \frac{1}{10(x+15)} \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$$

$$= \frac{1}{10(x+15)} \left[200x + \frac{8000}{3} - 50x - 1000/3\right]$$

$$= \frac{1}{x+15} \left[15x + 700/3\right]$$

(ii) $P(Y > 1/3\, X + 10)$

$$f(y) = 3/6875 \int_{6}^{5} x^2 + xy\ dx$$

$$= 3/6875 \left[\frac{x^3}{3} + x^2y/2\right]_0^5$$

$$= F(y) = \frac{3}{6875}\left[\frac{125}{3} + \frac{25}{2}y\right]$$

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$$= \frac{3}{6875} \int_{\frac{1}{3}x+10}^{20} \left(\frac{125}{3} + \frac{25}{2} y \right) dy$$

$$= \frac{3}{6075} \left[\frac{125y}{3} + \frac{25y^2}{4} \right]_{\frac{1}{3}x+10}^{20}$$

$$= \frac{3}{6875} \left[\frac{125}{3}(20) + \frac{25}{4}(400) - \frac{125}{3}\left(\frac{1}{3}x+10\right) - \frac{25}{4}\left(\frac{1}{3}x+10\right)^2 \right]$$

$$= \frac{3}{6875} \left(\frac{2500}{3} + 2500 - \frac{125x}{9} - \frac{1250}{3} \right)$$

$$- \frac{25}{4} \left[\frac{1}{9}x^2 + 100 + \frac{2x}{3} \right]$$

(6) $X \sim Poi(\lambda)$

$Y \sim Poi(\mu)$

$$M_X(t) = e^{\lambda(e^t-1)}$$

$$M_Y(t) = e^{\mu(e^t-1)}$$

$$Z = X - Y$$

$$M_Z(t) = \mathcal{E}(e^{tZ})$$
$$= \mathcal{E}(e^{t(X-Y)})$$
$$= \mathcal{E}(e^{tX} \cdot e^{-tY})$$

$$= \mathcal{E}(e^{tX}) \times \mathcal{E}(e^{-tY})$$

$$= \frac{\mathcal{E}(e^{tX})}{\mathcal{E}(e^{tY})}$$

$$= \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$$

$$= e^{(\lambda-\mu)(e^t-1)}$$

$$= e^{(\lambda-\mu)(e^t-1)} \sim Pois(\lambda-\mu)$$

- Let $Z = X+Y$

$$\mathcal{E}(e^{tZ}) = e^{\lambda(e^t-1)} \times e^{\mu(e^t-1)}$$

$$= e^{(\lambda+\mu)(e^t-1)}$$

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$\therefore z \sim Poi(\lambda + 4)$

(7)

i) $v(x/y=2) = \mathcal{E}(x^2/y=2) - [\mathcal{E}(x/y=2)]^2$

$$\mathcal{E}(x^2/y=2) = \cancel{\mathcal{E}x^2}$$

$$\sum x^2 \frac{p(x=x) \quad y=2}{p(y=2)}$$

$= x=1 + x=2 + x=3 + x=4$

$$= (1)^2 \frac{p(x=1, y=2)}{p(y=2)}$$

$$+ \frac{(2)^2 (p(x=2, y=2))}{p(y=2)}$$

$$+ (3)^2 \frac{p(x=3, y=2)}{p(y=2)}$$

$$+ (4)^2 \frac{p(x=4, y=3)}{p(y=2)}$$

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$$= \frac{P}{0.6} + \frac{4(0.3)}{0.6} + \frac{0}{0.6} +$$

$$\cancel{16.2(0} \quad \frac{16(0.2)}{0.6}$$

$$= \underline{\underline{8.833}}$$

$\rightarrow$ $E(x/y=2)$

$$= \frac{x\,(P(x=x, y=2)}{P(y=2)}$$

$$= \frac{1.0}{0.6} + \frac{2(0.3)}{0.6} + \frac{3(0.1)}{0.6} + \frac{4(0.2)}{0.6}$$

$$= \underline{\underline{2.833}}$$

ii) $f(u, v)$

$E(v/u=v)$

$$\frac{f(u,v)}{f(v)} = f(u/v) = v$$

$$f(v) = \frac{48}{67}\int_0^1 (2uv - u^2)\,du$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right]$$

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$= \frac{16}{67}\left[3v-1\right]$

$= f(v) = \frac{16}{67}\left[3v-1\right]$

$= \frac{f(u,v)}{f(v)} = \frac{\cancel{48}}{\cancel{67}} \frac{(2uv^2-u^2)}{\times \cancel{16}(3v-1)} \times \cancel{67}.$

$= \frac{3}{3v-1}(2uv^2-u^2) \cdot 2u^2v^2$

$= E(u/v=v) = \frac{3}{3v-1}\int_0^1 u\ (2uv^2-u)du$

$= \frac{3}{3v-1}\left[\frac{2u^3v^3}{3} - \frac{u^4}{4}\right]_0^1$

$= \frac{3}{3v-1}\left[\frac{2v^3}{3} - \frac{1}{4}\right]$

$= \frac{3}{3v-1}\left[\frac{-8v^2-3}{12}\right]$

$= \frac{8v^2-3}{4(3v-1)}$

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(8) $X \sim$ Gamma $(3, 2)$

$E(Y/x = x) = 3x + 1$

$V(Y/X = x) = 2x^2 + 5$

$E(Y) = E(E\,Y/x = x)$

$= (-(-3x+1))$

$= 3(-(x)+1$

$= 3(3/2)+1$

$= 9/2 + 1/1 = 9+2/2$

$V(Y) = E(V(X/Y)) + V((-X/Y)$

$= E(2x^2+5) + V(3x+11)$.

$= 2E(x^2) + 5 + 9V(x)$

$V(X) = E(x^2) - [E(x)]^2$

$= \frac{3}{4} + \frac{9}{4} = E(x^2)$

$= 12/4 = \underline{3}$

$V(Y) = 2(3)+5 + 9(3/4)$
$= 11 + 27/4 = \underline{71/4}$

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(9) $E(X) = 3$, $V(X) - 4$
$E(Y) = 4$, $V(Y) = 1$

$cov(X, Y) = -0.3\sqrt{4} = -0.6$

$Z = X + Y$.

i) $cov(X, X+Y) = cov(X,X) + cov(X,Y)$
$= V(X) + cov(X,Y)$
$= 4 + (-0.6)$
$= \underline{3.7}$

ii) $Var(Z) = cov(X+Y, X+Y)$
$= cov(X,X) + 2cov(X,Y) + cov(Y,Y)$
$= V(X) + 2cov(X,Y) + V(Y)$
$= 4 + 2(-0.6) + 1$
$= 5 - 1.2$
$= \underline{3.8}$

(10) $$K \int_1^\infty \int_1^\infty x^{-\alpha} e^{-y/\beta}\, dx\, dy = 1$$

$$K \int e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1}\right]_1^\infty = 1$$

$$\frac{K}{1-\alpha} \int e^{-y/\beta} [-1]$$

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$$= \frac{k}{\alpha - 1} \left[\frac{e^{-y/\beta}}{-1/\beta} \right]^{\infty} -$$

$$\frac{k}{\alpha - 1} (\beta) \left[-e^{-1/\beta} \right] = 1$$

$$= k = \frac{\alpha - 1}{(\beta)(e^{-1/\beta})}$$

② $N \sim \text{Neg.Bin}(k, p)$

$$P(N = n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$$

$$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$$

So, $k = 3$, $p = 0.9$

$X \sim \text{Gamma}(6, 2)$

$Y \to$ Total claim amount

MGF of Y:

$$M_y(t) = E(e^{yt} / n = N)$$

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$=$

$E(e^{yt}/N=n) =$

$E(e^{tx_1}) \cdot E(e^{tx_2}) \cdots E(e^{tx_n})$

$= \left(1 - \frac{t}{2}\right)^{-6n}$

So, $E(E(e^{yt}/N=n) =$

$E\left(\left(1 - \frac{t}{2}\right)^{-6n}\right)$

$= E\left(e^{N\log(1-t/2)^{-6}}\right)$

$= M_n\left(\log\left(1 - t/2\right)^{-6}\right)$

$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^K$

$= \left(\frac{pe^{\log(1-t/2)^{-6}}}{1-qe^{\log(1-t/2)^{-6}}}\right)^K$

$V(Y) = E(N)V(X) + (E(X))^2 \cdot V(N)$

$= \left(\frac{3}{0.9} \cdot \frac{6}{4}\right) + \left(\frac{6}{2} \cdot \frac{3(0.1)}{(0.9)^2}\right)$