PROBABILITY & STATISTICS

② $N \sim$ Neg Binomial (K, P)

$$P(N=n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$$

$$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$$

$K = 3$, $p = 0.9$

$$M_y t = M_n (\log M_x t)$$

$X \sim \text{Gamma}(6, 2)$

$Y \rightarrow$ total claim amount

(i) MGF OF Y

$$M_y(t) = E(E\,e^{yt}/N=n) \quad \text{— ①}$$

$$E(e^{yt}/N=n) = E(e^{x_1t + x_2t + \cdots x_nt})$$

$$= E(e^{x_1t}) \cdot E(e^{tx_2}) \cdots E(e^{nt})$$

$$= \left(1 - \frac{t}{2}\right)^{-6n} \quad \text{— ②}$$

comparing ① & ②,

$$E(E\,e^{yt}/N=n) = E\left(\left(1-\frac{t}{2}\right)^{-6n}\right)$$

$$= E\left(e^{N\log(1-1t/2)^{-6}}\right)$$

$$= M_n\left(\log\left(1-t/2\right)^{-6}\right)$$

$$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k \Rightarrow \left(\frac{pe^{\log(1-t/2)^{-6}}}{1-qe^{\log(1-t/2)^{-6}}}\right)^k$$

$$\Rightarrow \left(\frac{(0.9)(1-t/2)^{-6}}{1-0.1(1-t/2)^{-6}} \right)^3$$

$$V(Y) = E(N)\,V(X) + [E(X)]^2\, V(N)$$

$$= \left(\frac{3}{0.9} \times \frac{6}{4}\right) + \left(\frac{6}{2}\right)^2 \times \frac{3 \times 0.1}{(0.9)^2}$$

Standard Deviation $= 2.887$

(3) $f(x) = \lambda e^{(-\lambda/x-\alpha)}$

$$MGF = E[e^{tx}] \Rightarrow \int_\alpha^\infty e^{tx}\lambda e^{-\lambda(x-\alpha)}\,dx$$

$$M_x t = \lambda \int_\alpha^\infty e^{tx} e^{-\lambda x} e^{\lambda\alpha}\,dx$$

$$= e^{\lambda\alpha}\int_\alpha^\infty e^{-(\lambda-t)x}\,dx$$

$$= \frac{e^{\alpha\lambda}\lambda}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_\alpha^\infty$$

$$= \frac{e^{\lambda\alpha}\lambda}{t-\lambda}\left[-e^{-(\lambda-t)\alpha}\right]$$

$$= \frac{e^{\lambda\alpha}\lambda}{\lambda-t} e^{-\lambda\alpha} e^{t\alpha}$$

$$= e^{t\alpha}\frac{\lambda}{\lambda-t}$$

11) $M_x t = \dfrac{\lambda e^{t\alpha}}{\lambda - t}$

$= \lambda e^{t\alpha} (\lambda - t)^{-1}$

$= \lambda (e^{t\alpha} (\lambda - t)^{-2} (1) + (\lambda - t)^{-1} e^{t\alpha} (\alpha)]]$

$= \lambda ((\lambda - t)^{-1} e^{t\alpha} (\alpha) + e^{t\alpha} (\lambda - t)^{-2}]$

$= \lambda e^{t\alpha} (\lambda - t)^{-2} [\alpha (\lambda - t) + 1]$

$= \lambda (\lambda)^{-2} [\alpha (\lambda) + 1]$

$\dfrac{1}{\lambda} [\lambda\alpha + 1]$

$\Rightarrow \dfrac{\lambda\alpha + 1}{\lambda}$

(4) $G_V(t) = \left(\dfrac{p}{1 - qt} \right)^K$

$G_V t = \Sigma t^V \times P(V)$

$P(V = 4) = \dfrac{\sqrt{K+4}}{\sqrt{K}\sqrt{5}} p^K . q^K .$

$= \dfrac{\sqrt{K+4}}{4!\,K!} p^K q^K .$

$= \dfrac{(K+3)!}{4!\,(K-1)!} p^K \cancel{q} (1-p)^K$

(5) if $(x, y) = ax(x+y)$ $\qquad 0 < x < 5, 10 < y < 20$

$P \left(Y > \dfrac{1}{3} \wedge + 10 \right)$

$$a\int_{10}^{20}\int_{0}^{5}(x^2+xy)\,dx\,dy = 1$$

$$\int_{10}^{20}\left(\frac{x^3}{3}+\frac{x^2y}{2}\right)\Bigg|_0^5 dx = \frac{1}{a}$$

$$\int_{10}^{20}\left(\frac{125}{3}+\frac{25}{2}y\right)dy = \frac{1}{a}$$

$$a\left(\frac{2500}{3}+2500-\frac{1250}{3}-625\right)=1$$

$$a\left[\frac{1250}{3}+1875\right]=1$$

$$a\left(\frac{6875}{3}\right)=1$$

$$a=\frac{3}{6875}$$

$$a=0.00043636$$

(6) $X \sim Poi(\lambda)$ $\quad\quad$ $Y \sim Poi(\gamma)$

$M_X(t)=e^{\lambda(e^t-1)}$ $\quad\quad$ $M_Y(t)=e^{\gamma(e^t-1)}$

$$Z = X-Y$$

$$M_Z(t)=E(e^{tZ}) = E[e^{t(X-Y)}] = E(e^{tX}\cdot e^{-tY})$$

$$= E[e^{tX}]\,E(e^{-tY}) \Rightarrow \frac{E(e^{tX})}{E(e^{tY})}$$

$$= \frac{e^{\lambda(e^t-1)}}{e^{\gamma(e^t-1)}}$$

$$= e^{(\lambda-\mu)(e^t-1)}$$

So, by MGF of $Z = \underline{e^{(\lambda-\mu)\,e^t-1}}$

let $Z = X+Y$

$$E(e^{tz}) = e^{\lambda(e^t-1)} \times e^{\mu(e^t-1)}$$
$$= e^{(\lambda+\mu)\,e^t-1}$$

$Z \sim Poi(\lambda+\mu)$

7)

			X		
		1	2	3	4
y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

i) $V(X/Y=2) = E(x^2/Y=2) - [E(x/Y=2)]^2$

$$E(X^2/Y=2] = \sum x^2 \frac{P(X=x, Y=2)}{P(Y=2)}$$

when $X=1$ $\quad x=2 \quad x=3 \quad x=4$

$$1\,\frac{P(X=1, Y=2)}{Y=2} + (2)^2\,\frac{P(X=2, Y=2)}{Y=2} + 9\,\frac{(X=3, Y=3)}{Y=3}$$

$$\frac{1(0)}{0.6} + \frac{4(0.3)}{0.6} + \frac{9(0.1)}{0.6} + \frac{16(0.2)}{0.6}$$

$$\frac{0.6+0.3+0.8}{0.6} \Rightarrow 2.83333$$

0.807

11) $f(u,v) = \frac{48}{67}(2uv - v^2)$, $0 < u < 1$; $\frac{u}{2} < v < 2$

$E(v/u=v) = ?$

$$\frac{f(u,v)}{f(v)} = f\, v/v=v$$

$$f(v) = \frac{48}{67}\int_0^1 (2uv - u^2)\,du$$

$$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right]$$

$$= \frac{16}{67}[3v-1]$$

$$f(v) = \frac{16}{67}(3v-1)$$

$$\frac{f(u,v)}{f(v)} = \frac{\cancel{48}^{3}(2uv^2 - v^2)}{\cancel{67}} \frac{\cancel{67}}{\times 1\cancel{6}(3v-1)}$$

$$= \frac{3(2uv^2 - u^2)}{3v-1} \qquad \cdot 2u^2v^2 - u^3$$

$$E(u/V=v) = \frac{3}{3v-1}\int_0^1 u(2uv^2 - u^2)\,dy$$

$$\frac{3}{3v-1}\left[\frac{2u^3v^2}{3} - \frac{v^4}{4}\right]_0^1 \Rightarrow \frac{3}{3v-1}\left[\frac{2v^2}{3} - \frac{1}{4}\right]$$

$$\frac{3}{3v-1}\left[\frac{8v^2-3}{124}\right] = \frac{8v^2-3}{4(3v-1)}$$

$X \sim \text{Gamma}(3,2)$ $\quad E(X) = \frac{3}{2}$ $\quad V(x) = \frac{3}{4}$

$E(Y/X=x) = 3x+1$

$Var(Y/X=x) = 2x^2+5$

$E(Y) = E[E(Y/X=x)] \Rightarrow E(3x+1)$

$3E(X)+1$

$3\left(\frac{3}{2}\right)+1 \Rightarrow \frac{9}{2}+1 \Rightarrow \frac{9+2}{2} = \frac{11}{2}$

$$v(4) = E(v(X/Y)) + v(E(X/4))$$
$$= E(2x^2+5) + v(3X+1)$$
$$= 2E(X^2) + 5 + 9v(X)$$
$$v(x) = E(X^2) - [E(X)]^2$$
$$= \frac{3}{4} + \frac{9}{4} = E(X^2)$$
$$\frac{12}{4} = E(X^2)$$
$$E(X^2) = 3$$
$$v(Y) = 2(3) + 5 + 9(3/4)$$
$$= 11 + \frac{27}{4}$$
$$= \frac{44+27}{4} = \frac{71}{4}$$

(9) $E(X) = 3$, $V(X) = 4$
$E(Y) = 4$, $V(Y) = 1$

correlation coefficent $= -0.3\sqrt{Y} \Rightarrow cov(X,Y)$
$= -0.6$

$Z = X+Y$

i) $cov(X, X+Y) = cov(X,X) + cov(X,Y)$
$= V(X) + cov(X,Y)$
$= 4 + (-0.6)$
$= 3.4$

(ii) $var(Z) = cov(X+Y, X+Y)$
$= cov(X,X) + 2cov(X,Y) + cov(Y,Y)$
$= V(X) + 2cov(X,Y) + V(Y)$
$= 4 + 2(-0.6) + 1$
$= 5 - 1.2$
$= 3.8$