Calculus - Assignment 2

UNIT - 3 & 4

Chapter - 5, 6, 7

1. $\int_{\frac{1}{50}}^{2} \frac{e^{\frac{2}{w}}}{w^2} dw$

Let $\frac{2}{w} = t$. At $w = \frac{1}{50}$, $t = 100$

$\frac{-2}{w^2} = \frac{dt}{dw}$ At $w = 2$, $t = 1$

$\Rightarrow \frac{dw}{w^2} = \frac{-dt}{2}$

$\Rightarrow -\frac{1}{2}\int_{100}^{1} e^t dt$

$\Rightarrow \frac{-[e^t]_{100}^{1}}{2}$

$\Rightarrow \frac{99}{2} = \underline{\underline{49.5}}$

2. $\int \frac{2t^3+1}{(t^4+2t)^3} dt$

Let $t^4+2t = x$

$4t^3+2 = \frac{dx}{dt}$

$\Rightarrow (2t^3+1)dt = \frac{dx}{2}$

$\Rightarrow \frac{1}{2}\int \frac{dx}{x^3}$

$\Rightarrow \boxed{\frac{-1}{4x^2} + C}$

3. $f'(x) = x^4+3x-9$

$\Rightarrow f(x) = \int f'(x)\,dx$

$= \int (x^4+3x-9)\,dx$

$= \boxed{\frac{x^5}{5} + \frac{3x^2}{2} - 9x + C}$

4. $f(x) = \begin{cases} 6, & x>1 \\ 3x^2, & x\leq 1 \end{cases}$

$\Rightarrow I = \int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} 3x^2\,dx + \int_{1}^{3} 6\,dx$

$= [x^3]_{-2}^{1} + 6[x]_1^3$

$= 1+8 + 6[2]$

$= \underline{\underline{27}}$

5. $\int 3(8y-1)e^{4y^2-y}\,dy$

let $4y^2-y = t$

$8y-1 = \frac{dt}{dy}$

$\Rightarrow (8y-1)\,dy = dt$

$\Rightarrow 3\int e^t\,dt$

$\Rightarrow 3e^t + C$

$\Rightarrow \boxed{3e^{4y^2-y} + C}$

6. $f''(x) = 15\sqrt{x} + 5x^3 + 6$; $f(1) = -\frac{5}{4}$, $f(4) = 404$

$f'(x) = \int (15\sqrt{x} + 5x^3 + 6)\, dx$

$= 15 \times \frac{2}{3}(x)^{\frac{3}{2}} + \frac{5x^4}{4} + 6x + C_1$

$= 10(x)^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + C_1$

$f(x) = \int f'(x)\, dx$

$= \int \left(10(x)^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + C_1\right) dx$

$f(x) = \cancel{10}^{2} \times \frac{2}{\cancel{5}} (x)^{\frac{5}{2}} + \frac{x^5}{4} + 3x^2 + C_1 x + C_2$

$= 4(x)^{\frac{5}{2}} + \frac{x^5}{4} + 3x^2 + C_1 x + C_2$

$\Rightarrow$ ~~$f(1) = -\frac{5}{4} = 4$~~

$f(1) = -\frac{5}{4} = 4 + \frac{1}{4} + 3 + C_1 + C_2$

$C_1 + C_2 = \frac{-5 - 29}{4}$

$C_1 + C_2 = -\frac{17}{2}$

$\Rightarrow 2C_1 + 2C_2 = -17 \longrightarrow (1)$

$f(4) = 404 = 4(32) + 256 + 48 + 4C_1 + C_2$

$\Rightarrow 4C_1 + C_2 = -432 + 404$

$\Rightarrow 4C_1 + C_2 = -28 \longrightarrow (2)$

Solving (1) & (2)

$\Rightarrow 2C_1 + 2C_2 = -17 \quad \times 2$

$\Rightarrow \underline{4C_1 + C_2 = \mp 20}$

$3C_2 = -24 \Rightarrow C_2 = -\frac{14}{3}$

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$$C_1 = -\frac{17}{2} + \frac{14}{3}$$

$$C_1 = \frac{-51+28}{6} = \frac{-23}{6}$$

Ans $\rightarrow$ $\boxed{f(x) = 4(x)^{\frac{5}{2}} + \frac{x^5}{4} + 3x^2 - \frac{23}{6}x - \frac{14}{3}}$

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8. $\frac{dr}{d\theta} = \frac{r^2}{\theta}$; $r(1) = 2$

$$\frac{dr}{r^2} = \frac{d\theta}{\theta}$$

Integrating both sides :

$$\int \frac{dr}{r^2} = \frac{d\theta}{\theta}$$

$$\Rightarrow -\frac{1}{r} = \log|\theta| + C \quad \text{--- (1)}$$

At $\theta = 1$, $r = 2$

Subs in (1)

$$\Rightarrow -\frac{1}{2} = \cancel{\log} \ 0 + C$$

$$\Rightarrow C = -\frac{1}{2}$$

~~Ans~~ $\Rightarrow -\frac{1}{r} = \log\theta - \frac{1}{2}$

$$\Rightarrow \boxed{\log\theta - \frac{1}{r} = \frac{1}{2}}$$

9. $\frac{dv}{dt} = 9.8 - 0.196\,v$

$$\frac{dv}{9.8 - 0.196\,V} = dt$$

$$\frac{1}{9.8}\int \frac{dv}{1 - 0.02V} = \int dt$$

$$\Rightarrow \frac{-1}{9.8 \times 0.02}\log(1 - 0.02v) = t + C$$

$$\rightarrow \cancel{\infty}\log(1 - 0.02V) = -0.196t + k \qquad [k = -0.196\,C]$$

Ans $\Rightarrow \boxed{(1 - 0.02v) = e^{-0.196t} \cdot e^{k}}$

10. $ty' + 2y = t^2 - t + 1 \quad ; \quad y(1) = \frac{1}{2}$

$\cancel{y' = \frac{t^2 - t + 1 - 2y}{t}}$

$\cancel{\frac{dy}{dt} = t - 1 + \frac{1}{t} - \frac{2y}{t}}$

$\cancel{\frac{dy}{dt}} + y' + \frac{2y}{t} = t - 1 + \frac{1}{t} \quad —\ (i)$

This is of the form

$y' + P(t)y = Q(t) \ ; \ P(t) = \frac{2}{t} \ ; \ Q(t) = t - 1 + \frac{1}{t}$

$\rightarrow IF = e^{\int P(t)\,dt}$

$= e^{\int \frac{2}{t} dt}$

$= e^{2\log t} = e^{\log t^2}$

$= t^2$

Multiplying (i) with IF

$\Rightarrow t^2 y' + 2ty = t^3 - t^2 + \cancel{\frac{t^2}{t}}\ t$

$\rightarrow \frac{d}{dt}(t^2 y) = t^3 - t^2 + t$

Integrating both sides :

$t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C$

$\Rightarrow y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + Ct^{-2}$

At $t = 1,\ y = \frac{1}{2}$

$\Rightarrow C = \cancel{\frac{1}{2}} - \frac{1}{4} + \frac{1}{3} - \cancel{\frac{1}{2}} = \frac{1}{12}$

Ans $\rightarrow$ $\boxed{y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{2}t^{-2}}$

11. $\dfrac{dy}{dx} = \dfrac{xy^3}{\sqrt{1+x^2}}$; $y(0) = -1$

$$\int \frac{dy}{y^3} = \int \frac{x\,dx}{\sqrt{1+x^2}}$$

$$\frac{-1}{2y^2} = \frac{1}{2}\left[2\sqrt{1+x^2}\right] + C$$

$$\Rightarrow \frac{-1}{2y^2} = \sqrt{1+x^2} + C$$

At $x=0,\ y=-1$

$$\Rightarrow \frac{-1}{2} = 1 + C$$

$$\Rightarrow C = -\frac{3}{2}$$

$$\Rightarrow \frac{-1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Ans $\Rightarrow \boxed{\sqrt{1+x^2} + \dfrac{1}{2y^2} = \dfrac{3}{2}}$

~~12.~~ $\cancel{f(x) = x^3 - 10x^2 + 6}$ ~~for~~ $\cancel{x = 3}$

~~Using Taylor series~~ :

$\cancel{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots}$

$\cancel{f(x) = [6 +}$

$\cancel{f(x) = x^3}$

$\cancel{f'(x) = 3x^2 - 20x}$

$\cancel{f'(0) =}$

12. $f(x) = x^3 - 10x^2 + 6$; for $x = 3$

Taylor series is given by:

$$f(x) = f(k) + (x-k) f'(k) + \frac{(x-k)^2}{2!} f''(k) + \frac{(x-k)^3}{3!} f'''(k) + \dots$$

$$f(k) = f(3) = 3^3 - 10.3^2 + 6$$
$$= -57$$

$$f'(k) = 3k^2 - 20k$$
$$f'(3) = 3.3^2 - 20.3 = 27 - 60 = -33$$

$$f''(k) = 6k - 20$$
$$f''(3) = 18 - 20 = -2$$

$$f'''(k) = 6$$
$$f'''(3) = 6$$

→ Taylor series:

$$\Rightarrow f(x) = -57 + (x-3)[-33] + \frac{(x-3)^2}{2}[-2] + \frac{(x-3)^3 \times 6}{3 \times 2}$$

Ans $\boxed{f(x) = -57 - 33(x-3) - (x-3)^2 + (x-3)^3}$

13. Maclaurin series for $(1+x)^{n}$

Maclaurin series is given by:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) \dots$$

$f(0) = 1$

$f'(x) = m(1+x)^{m-1}$

$f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2}$

$f''(0) = m(m-1)$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$

$f'''(0) = m(m-1)(m-2)$

~~Similarly, $f^{m}(x) = m(m-1)(m-2)\ldots$~~

Ans $\rightarrow$ $f(x) = (1+x)^m = 1 + xm + \frac{x^2}{2}m(m-1) + \frac{x^3}{6}m(m-1)(m-2) + \frac{x^4}{24}m(m-1)(m-2)(m-3) + \ldots\ldots$

14. Maclaurin series for $f(x) = \sqrt{1+x}$

$f(0) = 1$

$f'(x) = \frac{1}{2\sqrt{1+x}}$; $f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2} \times \left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}} = \frac{-1}{4(1+x)^{\frac{3}{2}}}$; $f''(0) = -\frac{1}{4}$

$f'''(x) = -\frac{1}{4} \times \left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}} = \frac{3}{8(1+x)^{\frac{5}{2}}} = \frac{3}{8}$

$f^{iv}(x) = \frac{3}{8}\left(-\frac{5}{2}\right)(1+x)^{-\frac{7}{2}} = \frac{-15}{16(1+x)^{\frac{7}{2}}} = -\frac{15}{16}$

Ans $\Rightarrow$ $f(x) = 1 + \frac{x}{2} + \frac{x^2}{2}\left(-\frac{1}{4}\right) + \frac{x^3}{6} \times \frac{3}{8} - \frac{x^4}{\cancel{24}_{8}}\left(\frac{\cancel{15}^{5}}{16}\right) \ldots$

Ans $f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} \ldots$

15. $f(x) = e^{-6x}$ about $x = -4$

Using Taylor Series:

$$f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!}f''(k) + \frac{(x-k)^3}{3!}f'''(k) + \ldots$$

Here $k = -4$

$\Rightarrow f(-4) = e^{24}$

$\Rightarrow f'(k) = -6e^{-6k}$; $f'(-4) = -6e^{24}$

$\cancel{f'(}$

$\Rightarrow f''(k) = 36e^{-6k}$; $f''(-4) = 36e^{24}$

$\Rightarrow f'''(k) = -216e^{-6k}$; $f'''(-4) = -216e^{24}$

$\Rightarrow f^{iv}(k) = 1296e^{-6k}$; $f^{iv}(-4) = 1296e^{24}$

$\Rightarrow \cancel{f(x) = e^{24} - 6e^{24} + 6^2e^{24} - 6^3e^{24} + 6^4e^{24} \ldots}$

$\cancel{= e^{24}[1 - 6 + 6^2 - 6^3 + 6^4 \ldots]}$

16. $\cancel{f(x) = \ln(3+4x)}$; ~~about $x=0$~~

~~Using Taylor Series:~~

$\cancel{f(x) = f(k) + x}$

$$\Rightarrow f(x) = e^{24} - 6(x+4)e^{24} + \frac{(x+4)^2 \times 36e^{24}}{2} - \frac{(x+4)^3 \times 216e^{24}}{6} + \frac{(x+4)^4 \times 1296e^{24}}{24} \ldots$$

Ans $\Rightarrow$ $\boxed{f(x) = e^{24} - 6(x+4)e^{24} + 18(x+4)^2e^{24} - 36(x+4)^3e^{24} + 54(x+4)^4e^{24} \ldots}$

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16. $f(x) = \ln(3+4x)$; about $x=0$

Using Taylor series:

$$f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!} f''(k) + \frac{(x-k)^3}{3!} f'''(k) \ldots$$

Here $k = 0$

$f(k) = \ln 3$

$f'(k) = \frac{4}{3+4k}$; $f'(0) = \frac{4}{3}$

$f''(k) = \frac{-16}{(3+4k)^2}$; $f''(0) = -\frac{16}{9}$

$$f'''(k) = \frac{-(-16)\cdot(4)(2)\cancel{(3+4k)}}{(3+4k)^{\cancel{4}\,3}}$$

$$= \frac{128}{(3+4k)^3} \; ; \; f'''(0) = \frac{128}{27}$$

$$\Rightarrow f(x) = \ln 3 + \frac{4x}{3} + \frac{x^2}{2}\left(\frac{-16}{9}\right) + \frac{x^3}{\cancel{6}_{3}}\left(\frac{\cancel{128}^{64}}{27}\right) \ldots$$

Ans $\boxed{f(x) = \ln 3 + \frac{4x}{3} - \frac{8x^2}{9} + \frac{64x^3}{81} \ldots}$

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