Institute of Actuaries of India

Subject CT8 – Financial Economics

October 2015 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:-

Let K be the forward price, q be the dividend declared at time t, r be the risk free rate and consider entering into the following two positions at time 0

Assume 'No arbitrage' holds. Short Selling is allowed. No transactions costs and taxes.

A: One long forward contract i.e. obligation to buy one share to the contract holder at time T for price K and invest an amount of $\mathrm{K}\,(1+\mathrm{r})^{-\mathrm{T}} + \mathrm{q}(1+\mathrm{r})^{(-\mathrm{t})}$ at risk free rate

B: buy one share at price S_0 and invest an amount of q at risk free rate at time t

At time T, the value of both the positions will be

A: $\mathrm{S}_\mathrm{T} - \mathrm{K} + \mathrm{K} + \mathrm{q}(1+\mathrm{r})^{(\mathrm{T}-\mathrm{t})}$

B: $\mathrm{S}_\mathrm{T} + \mathrm{q}(1+\mathrm{r})^{(\mathrm{T}-\mathrm{t})}$

It can be seen that the cashflows at time T under both the positions are equal hence under no arbitrage condition, the value at time 0 of both these portfolios must be equal.

Hence $\mathrm{K}\,(1+\mathrm{r})^{-\mathrm{T}} + \mathrm{q}(1+\mathrm{r})^{(-\mathrm{t})} = \mathrm{S}_0$

i.e. $\mathrm{K} = \mathrm{S}_0(1+\mathrm{r})^{\mathrm{T}} - \mathrm{q}(1+\mathrm{r})^{(\mathrm{T}-\mathrm{t})}$ is the fair price of a forward contract on the given dividend paying share **[6 Marks]**

Solution 2:-

i) Suppose at any time t, 0<=t<T, an investor holds the following portfolio:

- Short one unit of derivative f contingent on S
- Long $\frac{\partial f}{\partial S}$ shares of the stock with price S_t, where S_t follows Geometric Brownian motion with drift μ and volatility σ. Wt is a standard Brownian motion
  $S_t = S_0 e^{\mu t - ½\sigma^2 t + \sigma W t}$
  The value of the portfolio $V(\mathrm{S}_\mathrm{t}, t) = -f(\mathrm{S}_\mathrm{t}, t) + \frac{\partial f}{\partial S}\mathrm{S}_\mathrm{t}$
  Market is assumed to be arbitrage free, risk free rate of interest is 'r'. Unlimited short selling i.e. negative holding allowed. Assets can be traded in infinitesimally small amounts. No taxes or transaction costs.

Using Ito's Lemma

$$df(\mathrm{S}_\mathrm{t}, \mathrm{t}) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}d\mathrm{S}_\mathrm{t} + 1/2\frac{\partial^2 f}{\partial S^2}(\mathrm{dS}_\mathrm{t})^2$$

$$-df(\mathrm{S}_\mathrm{t}, \mathrm{t}) + \frac{\partial f}{\partial S}d\mathrm{S}_\mathrm{t} = -\frac{\partial f}{\partial t}dt - 1/2\frac{\partial^2 f}{\partial S^2}(\mathrm{dS}_\mathrm{t})^2$$

$dS_t = \mu S_t dt + \sigma S_t dW_t$ gives us $(dS_t)^2 = \sigma^2 S_t^2 dt$

$-df(S_t, t) + \frac{\partial f}{\partial S} dS_t = -\frac{\partial f}{\partial t} dt - 1/2 \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 dt$

This expression involves dt but is independent of dW_t implying the investment return over a short interval of time is deterministic assuming that the market is arbitrage free. This means

$-df(St, t) + \frac{\partial f}{\partial S} ds = rV(St, t)\, dt$

So that $-\frac{\partial f}{\partial t} dt - \frac{1}{2\frac{\partial^2 f}{\partial S^2}} \sigma^2 S_t^2 dt = r(-f(St, t) + \frac{\partial f}{\partial S} St)\, dt$

Which implies $-\frac{\partial f}{\partial t} - 1/2 \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 = r(- f(St, t) + \frac{\partial f}{\partial S} St)$

Which implies $rf(St, t) = \frac{\partial f}{\partial t} + 1/2 \frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 + r\frac{\partial f}{\partial S} St$

i.e. $rf = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} \sigma^2 S_t^2 + r\frac{\partial f}{\partial S} St$ [5]

ii)

$$S_t = S_0 e^{\mu t - ½\sigma^2 t + \sigma Wt}$$

The expected growth rate of the stock is μ, which becomes r in risk-neutral world.

ln $S_t \sim N(\ln S_0 + \mu t - ½\sigma^2 t, \sigma^2 t)$

This means at time t, E (ln S_T) = lnS_t + $(\mu - ½\sigma^2)(T - t)$

In risk neutral world, the expected value is lnS_t + $(r - ½\sigma^2)(T - t)$

Using risk neutral valuation, the value/price of the security at time t is

$e^{-r(T-t)}[\ln S_t + (r - ½\sigma^2)(T - t)]$ [3]

iii) f = $e^{-r(T-t)}[\ln S_t + (r - ½\sigma^2)(T - t)]$

$\frac{\partial f}{\partial t} = r\, e^{-r(T-t)}[\ln S_t + (r - ½\sigma^2)(T - t)] - e^{-r(T-t)}(r - ½\sigma^2)$

$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$ and $\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2}$

Substituting the individual expressions in the right hand side of the equation we have,
$e^{-r(T-t)}[r \ln S_t - r(r - ½\sigma^2)(T-t) - ((r - ½\sigma^2) - ½\sigma^2 + r]$

=r $e^{-r(T-t)}[\ln S_t + (r - ½\sigma^2)(T - t)]$ = rf [3]

iv) The Portfolio will have PDE given by

$$r\pi = \frac{\partial\pi}{\partial t} + \frac{1}{2}\frac{\partial^2\pi}{\partial S^2}\sigma^2 S_t^2 + r\frac{\partial\pi}{\partial S}St$$

Since $\frac{\partial\pi}{\partial t} = \theta, \frac{\partial\pi}{\partial S} = \Delta, \frac{\partial^2\pi}{\partial S^2} = \Gamma$, it follows that $r\pi = \theta + r\,\Delta S + 1/2\Gamma\,\sigma^2 S_t^2$

For a delta neutral portfolio, $\Delta = 0$, and $\theta + 1/2\Gamma\,\sigma^2 S_t^2 = r\pi$

This shows that if θ is large and positive, Γ tends to be large and negative

The value of portfolio is independent of other Greeks ρ, λ, ν

If the delta and gamma of a portfolio are both zero then θ is the risk-free rate of growth of the portfolio. [2+1+1=4]

[15 Marks]

Solution 3:-

i)

S = 300
R =6%
σ =30%
K = 300
T =2/12 = 0.167

Price of a put option p	$Ke^{-rt}N(-d2) - S_0N(-d1)$
d1	$\frac{\left(ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right)}{\sigma T^{0.5}} = 0.1429$
d2	$\frac{\left(ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T\right)}{\sigma T^{0.5}} = 0.0204$
N(d1)	0.5568
N(d2)	0.5081
Price of a put option	13.132

[3]

ii)

The appropriate value of S becomes $S - 9 \times exp(-1/12 \times r)$ = 291.04

S = 291.04

R =6%
σ =30%
K = 300
T =2/12 = 0.167

Price of a put option p	$Ke^{-rt}N(-d2) - S_0N(-d1)$
d1	$\dfrac{\left(ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right)}{\sigma T^{0.5}} = -0.1046$
d2	$\dfrac{\left(ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T\right)}{\sigma T^{0.5}} = -0.2270$
N(d1)	0.4584
N(d2)	0.4102
Price of a put option	17.53

[3]

iii)

Parameters and value

S = K =300

r = 6%

T-t = 2/12 = 0.1667

u = exp($\sigma\sqrt{T-t}$)	1.09046
d = 1/u	0.91704
p	$\dfrac{(exp(r\delta t) - d)}{(u - d)} = 0.5073$
q = 1-p	0.4927

Payoff diagram

Pay off @ K=300 given in **bold**

t=0	t=1	t=2
		356.73
		0
	327.14	-----------
	0	
300.00		300
		0
	275.11	-----------
	24.89	-----------
		252.29
		47.71

Value of option = $47.71 \times q^2 \times \exp\big(-r(T-t)\big) = 11.43$ [5]

[11 Marks]

Solution 4:-

i) Similarities:
Both are one factor models.
Both have drifts that are deterministic
Both show some degree of local mean reversion

Differences

Black-Karasinski Model	Vasicek Model
Model given by $d(\ln r(t)) = k(t)(\theta(t) - \ln r(t))dt - \sigma(t)dW(t)$	$dr(t) = \alpha\big(\mu - r(t)\big)dt - \sigma dW(t)$
ln r can take any value but r will always be positive	**The interest rates can go negative as per this model**
Local mean reversion of ln r. Hence r/r_0 shows local mean reversion.	**Local mean reversion of the interest rates**
Time- heterogeneous model	Time-homogenous model
Speed of mean reversion is time dependent	Speed of mean reversion is constant
Complex to model (in comparison to Vasicek Model)	Very easy to model

[5]

ii) ln(r) follows a normal distribution hence 'r' follows a log-normal distribution…
[1]

[6 Marks]

Solution 5:-

i) Mental accounting (person is keeping separate account of a particular stock and intends to earn on it instead of looking at the overall earnings of his asset portfolio)
Assigning probabilities (as the person is 'hoping' for a favourable event and hence assigning lower probability to the undesired losses) [1]

ii) A. Prospect theory (necessary),
B. Assigning probabilities (dislike for valence outcomes), Status quo bias, over confidence [2]

iii) Effect of options
A. Applicable for A-Status quo bias
B. Applicable for A Regret aversion
C. Applicable for B Ambiguity aversion
D. Applicable for C Recency Effect [4]

[7 Marks]

Solution 6:-

i)

B = Bond price = $e^{-y(T-t)}$

$\frac{\partial B}{\partial t} = ye^{-y(T-t)} = yB$

$\frac{\partial B}{\partial y} = -(T-t)e^{-y(T-t)} = -(T-t)B$

$\frac{\partial^2 B}{\partial^2 y} = (T-t)^2 B$

$dB = [\frac{\partial B}{\partial y} dy + \frac{1}{2\frac{\partial^2 B}{\partial^2 y} dy^2} + \frac{\partial B}{\partial t} dt]$

$= \frac{\partial B}{\partial y}[\mathrm{a}(y_0 - \mathrm{y})\mathrm{dt} + \mathrm{sydw}] + \frac{\partial B}{\partial t} dt + 1/2 \frac{\partial^2 B}{\partial^2 y}[\mathrm{a}(y_0 - \mathrm{y})\mathrm{dt} + \mathrm{sydw}]\text{^}2$

$=[\frac{\partial B}{\partial y}\mathrm{a}(y_0 - \mathrm{y}) + \frac{\partial B}{\partial t} + 1/2 \frac{\partial^2 B}{\partial^2 y} s^2 y^2]dt + \frac{\partial B}{\partial y}$ sydw by Ito's Lemma

Substituting the partial derivatives above we have

$dB = [-\mathrm{a}(y_0 - \mathrm{y})(\mathrm{T}-\mathrm{t}) + y + 1/2 s^2 y^2 (T-t)^2]\mathrm{Bdt} - \mathrm{sy}(\mathrm{T}-\mathrm{t})\mathrm{Bdw}$

The question does not specify that 'y' is yield however is obvious that it is not the process for bond price as the question (i) is about deriving the process for bond price.

All candidates are given appropriate marks for a reasonable attempt even if y is assumed to be the bond price. [5]

ii)

$$\frac{\partial B}{\partial y} = -(T-t)e^{-y(T-t)} = -(T-t)B$$

which shows that as t tends to T, $\frac{\partial B}{\partial y}$ tends to zero driving volatility to zero [1]

iii)

$y = \frac{e^{r(T-t)}}{S}$ which means dy=$\frac{\partial y}{\partial S} dS + 1/2 \frac{\partial^2 y}{\partial^2 S} dS^2 + \frac{\partial y}{\partial t} dt$

or dy $= -\frac{e^{r(T-t)}}{S^2}[\mu S dt + \sigma S dw] + 1/2(2)\frac{e^{r(T-t)}}{S^3}[\mu S dt + \sigma S dw]^{\,2} - \frac{e^{r(T-t)}}{S} dt$

$= -\frac{e^{r(T-t)}}{S}[\mu dt + \sigma dw] + \frac{e^{r(T-t)}}{S}\sigma^2 dt - \frac{e^{r(T-t)}}{S} dt$

$= -y[\mu dt + \sigma dw] + y\sigma^2 S^2 dt - y dt$

$= -y[\mu + 1 - \sigma^2]dt - y\,\sigma dw$

which shows that the process is a martingale with drift zero if $\mu = \sigma^2 - 1$ [4]

[10 Marks]

Solution 7:-

i)

The value of the risk free zero coupon bond with a principal of Rs. P is Pe^{-yrT} whereas the value of the similar corporate bond is Pe^{-yT}

The expected value of the bond is given by

$$q \times 0 + (1-q)Pe^{-yrT} = P(1-q)e^{-yrT}$$

The price of bond with yield 'y' is Pe^{-yT}

So that equating the two expressions we have

$$Pe^{-yT} = P(1-q)e^{-yrT}$$

or $q = 1 - e^{-[y-yr]T}$ [4]

ii) The expected loss from default is $\mathbf{P}(e^{-yrT} - e^{-yT})$ [1]

iii) The above equation for the expected value of the bond becomes
$qPRe^{-yrT} + (1-q)\,P\,e^{-yrT}$

so that we need to equate the two expressions:

$$qPRe^{-yrT} + (1-q)Pe^{-yrT} = Pe^{-yT}$$

$q = [e^{-yrT} - e^{-yT}]/[(1-R)\,e^{-yrT}] = [1 - e^{-[y-yr]T}]/(1-R)$ [3]

[8 Marks]

Solution 8:-

i) Let x_n and x_v be the proportion of Narrow plc & Volatile plc respectively in Fund A.

The expected return of the fund E_A is given by $E_A = x_N E_N + x_V E_V$

The fund's variance is given by $\sigma_A^2 = x_N^2\sigma_N^2 + x_V^2\sigma_V^2 + 2\,x_N\,x_V\,\sigma_N\,\sigma_V\rho_{NV}$

Substituting $x_N = 1 - x_V$

$$\sigma_A^2 = (1-x_V)^2\sigma_N^2 + x_V^2\sigma_V^2 + 2(1-x_V)x_V\,\sigma_N\,\sigma_V\rho_{NV}$$

Taking derivatives w.r.t x_V and setting it to zero:

$$\frac{\partial\sigma_A^2}{\partial x_V} = -2(1-x_V)\sigma_N^2 + 2x_V\sigma_V^2 + 2(1-2x_V)\,\sigma_N\,\sigma_V\rho_{NV} = 0$$

$$x_V = \frac{\sigma_N^2 - \sigma_N\,\sigma_V\rho_{NV}}{\sigma_V^2 - 2\,\sigma_N\,\sigma_V\rho_{NV} + \sigma_N^2}$$

From the given table, Expected return & Variance of N and V are.

	Narrow	Volatile
$E = \sum p_i E_i$	2.85%	1.40%
$V = \sum E_i^2 - E^2$	0.0014	0.0065

Substituting the above value in equation for x_V, x_V =10.42% [5]

ii)

	Logsystems plc	ProLog plc
$E = e^{(\mu+0.5\sigma^2)}$	12.06%	9.02%
$V = \left(e^{\sigma^2}-1\right)\left(e^{2\mu+\sigma^2}\right)$	0.03234	0.0239

$$E_B = x_L E_L + x_P E_P = 40\% \times 12.06\% + 60\% \times 9.02\% = 10.24\%$$

$$\sigma_B^2 = x_L^2\sigma_L^2 + x_P^2\sigma_P^2 + 2\, x_L\, x_P\, \sigma_L\, \sigma_P \rho_{LP}$$
$$= 0.4^2 \times 0.0323\%\% + 0.6^2 \times 0.0239\%\% + 2 \times 0.4 \times 0.6 \times \sqrt{0.0323\%\% \times 0.0239\%\%} \times (-0.75)$$
$$= 0.00377\%\%$$ [4]
$$\sigma_B = 0.064\%$$

[9 Marks]

Solution 9:-

i) It will pay to add stock B to fund F with a positive weight because:

- As B has higher expected return, Inclusion of B will possibly increase the expected returns on the portfolio.
- As B is uncorrelated with the current portfolio the diversification will reduce the variance of the overall portfolio.

[2]

ii) If the new portfolio is P then,

$E_P = x_B E_B + x_F E_F = 0.4 \times 19\% + 0.6 \times 14\% = 16.0\%$

$\sigma_P^2 = x_B^2\sigma_B^2 + x_F^2\sigma_F^2 + 2\, x_B\, x_F\, \sigma_B\, \sigma_F \rho_{BF}$

$= 0.4^2 \times 0.6^2 + 0.6^2 \times 0.2^2 + 2\, x_B\, x_F\, \sigma_B\, \sigma_F \times 0 = 0.072\%\%$

$\beta_P = \frac{Cov(B,P)}{\sigma_P^2} = \frac{0.4\times 0.6^2}{0.072} = 2$

Risk Adjusted Return = $5.5\% + 2(16\% - 5.5\%) = 26.5\%$

The risk adjusted return is not equal to the return on stock B and hence the colleague is right. [3]

iii) Repeat the above calculations with weight of B as 15%

$E_P = x_B E_B + x_F E_F = 0.15 \times 19\% + 0.85 \times 14\% = 14.75\%$

(0.5)

$\sigma_P^2 = x_B^2\sigma_B^2 + x_F^2\sigma_F^2 + 2\, x_B\, x_F\, \sigma_B\, \sigma_F \rho_{BF}$

$= 0.15^2 \times 0.6^2 + 0.85^2 \times 0.2^2 + 2\, x_B\, x_F\, \sigma_B\, \sigma_F \times 0 = 0.037\%\%$

$\beta_P = \frac{Cov(B,P)}{\sigma_P^2} = \frac{0.15\times 0.6^2}{0.037} = 1.46$

Risk Adjusted Return = $5.5\% + 1.46(14.75\% - 5.5\%) = 19.0\%$

The risk adjusted return is equal to the return on stock B and hence this weight is appropriate for the intended purpose of achieving returns same as that on stock B.

[3]

iv)

Expected returns of Fund F: $14 = 7I_1 + 2I_2 - 2$

$16 = 7I_1 + 2I_2$ … Equ A

Expected returns of Stock B: $19 = 10I_1 + 1I_2 + 4.5$

$14.5 = 10I_1 + 1I_2$ .. Equ B

Multiply Equ B by 2 and subtract Equ A from it.

$$29 - 16 = (20 - 7)I_1 + (2 - 2)I_2$$

$13 = 13I_1$ Hence $I_1 = 1$

Substitute value of I_1 in Equ B, $14.5 = 10(1) + 1I_2$ Hence $I_2 = 4.5$

When I_1 =1.7 and I_2 = 4,

$E_F = 7(1.7) + 2(4) - 2 = 17.9$ and

$E_B = 10(1.7) + 1(4) + 4.5 = 25.5$

If proportion of Stock B in the overall portfolio is 'z' then

$E_P = (1 - z)(17.9) + z(25.5) = 17.9 + 7.6z$

For $E_P = 19$

$19 = 17.9 + 7.6z$ Hence $z = \frac{1.1}{7.6} = 0.1447$

Z=14.47% [6]

[14 Marks]

Solution 10:-

i)

a) Variance – Quadratic utility function [1]

b) Down-side semi variance – Liner above mean, quadratic below the mean [2]

c) State dependent function with a discontinuity at the returns specified at the 97.5th percentile. [2]

ii) Skewness is the appropriate risk measure if investor's utility function is best described by a cubic equation. [1]

iii) The choice of an investor as regards to selecting between two portfolios will depend on which portfolio stochastically dominates the other.

Investor A

$U_A(w) = w + 1$

$U'_A(w) = 1 > 0$
Hence the investor prefers more to less
$U''_A(w) = 0$
Hence the investor is risk neutral

Investor B
$U_B(w) = w^{0.5}$
$U'_B(w) = 0.5w^{-0.5}$ This is always > 0 as 'w' is always >0
Hence the investor prefers more to less
$U''_B(w) = -0.25w^{-1.5}$ This is always <0 as 'w' is always >0
Hence the investor is risk averse

	Cumulative probability	
Return	Portfolio U	Portfolio V
6%	0.25	0
7%	0.50	0.75
8%	0.75	0.75
9%	1.00	1.00

For Portfolio U, probability of receiving any amount L or less that is never greater, and sometimes strictly less, than that offered by Portfolio V. Hence portfolio V (first order) stochastically dominates the Portfolio U.

	Sum of Cumulative probability	
Return	Portfolio U	Portfolio V
6%	0.25	0
7%	0.75	0.75
8%	1.50	1.50
9%	2.50	2.50

Portfolio V (second-order stochastically) dominates U because the sum of its cumulative probabilities is never greater than that of U and for one outcome is strictly less.

Choice of Portfolio

As investor A is risk neutral we need to check the First order stochastic dominance.

As Portfolio V (first order) dominates the portfolio U, investor A is likely to choose portfolio V for investment.

As investor B is risk averse we need to check the Second order stochastic dominance.

As Portfolio V (second order) dominates the portfolio U, investor B is also likely to choose portfolio V for investment.

[8]

[14 Marks]

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